Discrete Mathematics: Combinatorics and Graph Theory

Homework 3 Solution

Instructions. Solve any 10 questions. Typeset or write neatly and show your work to receive full credit.

1. Find a formula for a_n given the stated recurrence relation and initial values:

   (a) $a_n = 3a_{n-1} - 2$ for $n \geq 1$ with $a_0 = 1$.
   $a_n = 1$ for $n \geq 0$, which is easily proved by induction. The base case is given in the initial condition. The inductive step assumes that $a_{n-1} = 1$ and computes $a_n = 3a_{n-1} - 2 = 3 \times 1 - 2 = 1$.

   (b) $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 2$ with $a_0 = 1$, $a_1 = 8$.
   Write the characteristic polynomial $x^2 - x - 2 = 0 \Rightarrow x = 2$ and $x = -1$. The general solution is $a_n = A(2)^n + B(-1)^n$. Applying initial conditions $1 = A + B$ and $8 = 2A - B$. Solving gives $A = 3$ and $B = -2$ for $a_n = 3(2)^n + -2(-1)^n$.

   (c) $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 2$ with $a_0 = a_1 = 1$.
   The characteristic equation is $x^2 - 2x - 3 = 0 \Rightarrow x = 3$ and $x = -1$. The general solution is $a_n = A(3)^n + B(-1)^n$. Applying initial conditions $1 = A + B$ and $1 = 3A - B$. Solving gives $A = 1/2$ and $B = 1/2$ for $a_n = [3n + (-1)n]/2$.

   (d) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$ with $a_0 = 1$, $a_1 = 3$.
   The characteristic equation is $x^2 - 5x + 6 = 0 \Rightarrow x = 3$ and $x = 2$. The general solution is $a_n = A(3)^n + B(2)^n$. Applying initial conditions $1 = A + B$ and $3 = 3A + 2B$. Solving gives $A = 1$ and $B = 0$ for $a_n = 3^n$.

   (e) $a_n = 3a_{n-1} - 1$ for $n \geq 1$ with $a_0 = 1$.
   The characteristic root is 3 and the inhomogenous term is a constant, so the solution has the form $A(3)^n + c$. For a particular solution involving the inhomogeneous term, we require $c = 3c - 1$ to obtain $c = 1/2$. Now the initial condition yields $1 = A \times 3^0 + 1/2$, so $A = 1/2$. Thus the solution is $a_n = (3n + 1)/2$.

2. Find a recurrence relation for the number of ternary strings of length n that contain either two consecutive 0s or two consecutive 1s.
   Let a_n denote the number of ternary strings with two consecutive 0's or two consecutive 1's. We could start with a 2, and follow with a string of two consecutive 0's or two consecutive 1's, which can be done in a_{n-1} ways. However, we for each k from 0 to $n-2$, the string could start with $n-1-k$ alternating 0's and 1's followed by a 2, and then be followed by a string of length k containing either two consecutive 0's or two consecutive 1's. There are $2a_k$ such strings, since there are two ways for the initial piece to alternate. The other possibility is that the string has no 2's at all. Then it must consist of $n-k-2$ alternating 0's and 1's, followed by a pair of 0's or 1's, followed by any string of length k. There are 2×3^k such strings. The sum of these as k runs from 0 to $n-2$ is $3^{n-1} - 1$. Combining terms we have the following recurrence relation for $n \geq 2 : a_n = a_{n-1} + 2a_{n-2} + 2a_{n-3} + \cdots + 2a_0 + 3^{n-1} - 1$. Substitute $n-1$ for n and subtract the above for a closed form solution $a_n = 2a_{n-1} + a_{n-2} + 2 \times 3^{n-2}$.

   (a) What are the initial conditions?
   $a_0 = a_1 = 0$.

   (b) How many ternary strings of length six contain two consecutive 0s or two consecutive 1s?

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\begin{aligned}
a_2 &= a_1 + 2a_0 + 3^1 - 1 = 0 + 2 \times 0 + 3 - 1 = 2 \\
a_3 &= a_2 + 2a_1 + 2a_0 + 3^2 - 1 = 2 + 2 \times 0 + 2 \times 0 + 9 - 1 = 10 \\
a_4 &= a_3 + 2a_2 + 2a_1 + 2a_0 + 3^3 - 1 = 10 + 2 \times 2 + 2 \times 0 + 2 \times 0 + 27 - 1 - 40 \\
a_5 &= a_4 + 2a_3 + 2a_2 + 2a_1 + 2a_0 + 3^4 - 1 = 40 + 2 \times 10 + 2 \times 2 + 2 \times 0 + 81 - 1 = 144 \\
a_6 &= a_5 + 2a_4 + 2a_3 + 2a_2 + 2a_1 + 2a_0 + 3^5 - 1 \\
&= 144 + 2 \times 40 + 2 \times 10 + 2 \times 2 + 2 \times 0 + 243 - 1 = 490
\end{aligned}
$$

There are 490 ternary strings of length 6 containing two consecutive 0's or two consecutive 1's.

3. A bus driver pays all tolls, using only nickels and dimes, by throwing one coin at a time into the mechanical toll collector.

   (a) Find a recurrence relation for the number of different ways the bus driver can pay a toll of n cents (where the order in which the coins are used matters).
   Let a_n be the number of different ways to pay a toll of n cents. Writing out a few examples is useful for intuition: $a_0 = 0, a_5 = 1, a_{10} = 2, a_{15} = 3$. Either the last coin is a dime or a nickel. If the last coin is a dime, then n coins can be paid a_{n-5} ways. If the last coin is a nickel then n coins can be paid a_{n-10} ways. Therefore $a_n = a_{n-5} + a_{n-10}$ for $n \geq 10$. We can only pay tolls corresponding to multiples of 5 so $a_{5n} = a_{5(n-1)} + a_{5(n-2)}$ for $n \geq 2$.

   (b) In how many different ways can the driver pay a toll of 45 cents?
   We need to compute a_{45} which requires initial conditions $a_0 = 1$ and $a_5 = 1$. Iterating we find that $a_{10} = 2, a_{15} = 3, a_{20} = 5, a_{25} = 8, a_{30} = 13, a_{35} = 21, a_{40} = 34, a_{45} = 55$.

4. Show that the Fibonacci numbers satisfy the recurrence relation $f_n = 5f_{n-4}+3f_{n-5}$ for $n = 5, 6, 7, \cdots$, together with the initial conditions $f_0 = 0, f_1 = 1, f_2 = 1, f_3 = 2$, and $f_4 = 3$. Use this recurrence relation to show that f_{5n} is divisible by 5, for $n = 1, 2, 3, \cdots$.
The original definition of Fiboniacci numbers is $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$ and $f_0 = 0,\ f_1 = 1$. Then $f_2 = 1, f_3 = 2, f_4 = 3$. Since $f_{n-3} = fn - 4 + f_{n-5},\ f_{n-2} = f_{n-3} + f_{n-4}$ and $f_{n-1} = f_{n-2} + f_{n-3}$ we have:

$$\begin{aligned} f_n &= f_{n-1} + f_{n-2} = (f_{n-2} + f_{n-3}) + f_{n-2} = 2f_{n-2} + f_{n-3} \\ &= 2(f_{n-3} + f_{n-4}) + f_{n-3} = 3f_{n-3} + 2f_{n-4} \\ &= 3(f_{n-4} + f_{n-5}) + 2f_{n-4} = 5f_{n-4} + 3f_{n-5} \end{aligned}$$

We will prove $5 \mid f_{5n}$ for all $n \geq 1$ via induction. The base case is true since $f_5 = f_3 + f_4 = 5$. The inductive hypothesis assumes that $P(n)$ is true so that $f_{5n} = 5k$ for some k. Then $f_{5(n+1)} = f_{5n+5} = 5f_{5(n+1)-4} + 3f_{5(n+1)-5} = 5f_{5n+1} + 3f_{5n} = 5f_{5n+1} + 15k = 5(f_{5n+1} + 3k)$ which is divisible by 5.

5. Solve the recurrence relation $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$ with $a_0 = -5, a_1 = 4$, and $a_2 = 88$.
The characteristic polynomial $x^3 - 6x^2 + 12x - 8 = (r - 2)^3 = 0$ has root 3 of multiplicity 3. The general solution is thus $a_n = \alpha_1 2^n + \alpha_2 n 2^n + \alpha_3 n^2 2^n$. To find the coefficients solve the system of equations:

$$\begin{aligned} a_0 &= -5 = \alpha_1 \\ a_1 &= 4 = 2\alpha_1 + 2\alpha_2 + 2\alpha_3 \\ a_2 &= 88 = 4\alpha_1 + 8\alpha_2 + 16\alpha_3 \end{aligned}$$

We find $\alpha_1 = -5, \alpha_2 = 1/2, \alpha_3 = 13/2$. Therefore $a_n = -5 \times 2^n + (1/2) \times n \times 2^n + (13/2) \times n^2 \times 2^n$.

6. What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has the roots $-1, -1, -1, 2, 2, 5, 5, 7$?

$$a_n = (\alpha_{1,0} + \alpha_{1,1} n + \alpha_{1,2} n^2)(-1)^n + (\alpha_{2,0} + \alpha_{2,1} n)2^n + (\alpha_{3,0} + \alpha_{3,1} n)5^n + \alpha_{4,0} 7^n$$

7. Find the solution of the recurrence relation $a_n = 2a_{n-1} + 3 \cdot 2^n$.
The associated homogeneous recurrence relation is $a_n = 2a_{n-1}$. Therefore $a_n^{(h)} = \alpha 2^n$. To solve the non-homogeneous part, look for a function of the form $a_n = cn \cdot 2^n$. Plug this into the recurrence to obtain $cn \cdot 2^n = 2c(n-1)2^{n-1} + 3 \cdot 2^n$. Divide by 2^{n-1} to obtain $2cn = 2c(n-1) + 6 \Rightarrow c = 3$. Therefore $a_n^{(p)} = 3n \cdot 2^n$. The general solution is the sum of the homogeneous solution and the particular solution $a_n = \alpha 2^n + 3n \cdot 2^b = (3n + \alpha)2^n$.

8. Find the solution of the recurrence relation $a_n = 4a_{n-1} - 3a_{n-2} + 2^n + n + 3$ with $a_0 = 1$ and $a_1 = 4$. The associated homogeneous recurrence relation is $a_n = 4a_{n-1} - 3a_{n-2}$. The characteristic equation is $x^2 - 4x + 3 = (x-1)(x-3) = 0 \Rightarrow x = 1$ and $x = 3$. Therefore $a_n^{(h)} = \alpha_1 1^n + \alpha_2 3^n$. Since the non-homogeneous part is an exponential and a linear term we can the particular solution in two parts $F_1(n) = 2^n$ and $F_2(n) = n + 3$.

   The particular solution for $F_1(n)$ is $a_n^{(p_1)} = C \cdot 2^n$. Plugging into the recurrence relation gives $C2^n = 4C2^{n-1} - 3C2^{n-2} + 2^n \Rightarrow C = -4$ and $a_n^{(p_1)} = -4 \cdot 2^n$.

   Consider $a_n^p = dn^2 + en + f$ and substitute in the remaining part of the recurrence:

$$\begin{aligned} dn^2 + en + f &= 4(d(n-1)^2 + e(n-1) + f) - 3(d(n-2)^2 + e(n-2) + f) + n + 3 \\ &\Rightarrow 3 = 4d + e + 1 \Rightarrow d = -1/4, \qquad -8d + 2e + f + 3 = f \\ &\Rightarrow e = -5/2\,,\ f = 3 \\ &\Rightarrow a_n^{(p)} = -\frac{1}{4}n^2 - \frac{5}{2}n + 3 \end{aligned}$$

   Therefore $a_n = \alpha_1 + \alpha_2 3^n - 4\cdot 2^n - \frac{n^2}{4} - \frac{5}{2}n + 3$. Applying the initial conditions $a_0 = 1 = \alpha_1 + \alpha_2 - 3 + 3 \Rightarrow \alpha_1 + \alpha_2 = 2$ and $a_1 = a_1 + 3\alpha_2 - 8 - 1/4 - 5/2 + 3 = 4 \Rightarrow \alpha_1 + 3\alpha_2 = 47/4$ and $\alpha_1 + \alpha_2 = 2$. Solving yields $\alpha_2 = 39/8$ and $\alpha_1 = -23/8$. Therefore

$$\begin{aligned} \alpha_n &= -\frac{23}{8} + \frac{39}{8}3^n - 4\cdot 2^n - \frac{n^2}{2} - \frac{5}{2} + 3 \\ \Rightarrow \alpha_n &= \frac{13}{8}\cdot 3^{n+1} - 4\cdot 2^n - \frac{n^2}{4} + \frac{1}{8} - \frac{5}{2}n \end{aligned}$$

9. Suppose $\langle a \rangle$ satisfies the recurrence $a_n = -a_{n-1} + \lambda^n$. Determine the values of λ such that $\langle a \rangle$ can be unbounded.
   Solving for the characteristic root of the homogeneous part yields -1. When $\lambda \neq -1$, a particular solution $b_n = \frac{1}{\lambda+1}n^{\lambda+1}$ is found by solving $C\lambda^n = -C\lambda^{n+1} + \lambda^n$ for C. The general solution is then $a_n = A(-1)^n + \frac{1}{\lambda+1}n^{\lambda+1}$. This is unbounded for $|\lambda| > 1$.

   When $\lambda = -1$, a particular solution $b_n = n\lambda^n$ is found by solving $Cn\lambda^n = -C(n-1)\lambda^{n-1} + \lambda^n$ for C after dividing by λ and equating corresponding coefficients. The linear term confirms that $\lambda = -1$ and the constant term yields $C = -\lambda = 1$. The general solution is then $a_n = A(-1)^n + n(-1)^n$ for constant A, which is again unbounded.

10. Let $a_n = n^3$. Find a constant-coefficient first-order linear recurrence relation satisfied by $\langle a \rangle$. Does there exist a homogeneous constant-coefficint first-order linear recurrence relation satisfied by $\langle a \rangle$? Why or why not?
    The solution has no nontrivial exponential part. We will look for a first-order relation of the form $a_n = a_{n-1} + f(n)$. Set $f(n) = n^3 - (n-1)^3 = 3n^2 - 3n + 1$.

    Every homogeneous constant-coefficinet first-order linear recurrence has the form $a_n = ca_{n-1}$, with general solution Ac^n. The constant A cannot be chosen to make $Ac^n = n^3$.

11. Derive a general formula for the recurrence $a_n = ca_{n-1} + f(n)\beta^n$ where f is a polynomial and β a constant.
    Define b_n by setting $a_n = \beta^n b_n$. Substituting into the recurrence for $\langle a \rangle$ and canceling β^n yields $b_n = (c/\beta)b_{n-1} + f(n)$. If $c \neq \beta$ then $b_n = Ac^n\beta^{-n} + p(n)$ where $p(n)$ is a polynomial of degree d. If $c = \beta$ then $b_n = p(n)$ where $p(n)$ is a polynomial of degree $d+1$. Multiplying by β^n yields $a_n = Ac^n + p(n)\beta^b$ and $a_n = p(n)\beta^b$ in the two cases.

12. Let f be a polynomial of degree n. The *first difference* of f is the function $g = \Delta f$ defined by $g(x) = f(x+1) - f(x)$. The *k-th difference* of f is the function $g^{(k)}$ defined inductively by $g^{(0)} = f$

and $g^{(k)} = \Delta g^{(k+1)}$ for $k \geq 1$. Obtain a formula for the nth difference of f.
This is analogous to the polynomial derivative result sketched in lecture with the Binomial theorem. The n^{th} difference of a polynomial of degree n is the constant $n!$ times the leading coefficient of the polynomial. We prove that the first difference of a polynomial of degree d with leading coefficient a is a polynomial of degree $d-1$ with leading coefficient da. Since the n^{th} difference is obtained by applying the first difference n times, this yields the claimed result.

By the definition of first difference, the first difference of a sum of polynomials is the sum of their first differences, and the first difference of c times a polynomial is $c\Delta f$. It therefore suffices to prove the claim for pure powers. We have $\Delta x^d = (x+1)^d - x^d$. Expanding $(x+1)^d$ by the Binomial Theorem shows that the result is a polynomial of degree $d-1$ with leading coeficient d.